Sample paths properties of Gaussian fields with equivalent spectral densities

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Abstract

We prove that if X and Y are two Gaussian fields with equivalent spectral densities, they have the same sample paths properties in any separable Banach space continuously embedded in $\mathcal{C}^0(K)$ where K is a compact set of $\mathbb{R}^d$.

Résumé

Propriétés des trajectoires de champs gaussiens ayant des densités spectrales équivalentes Nous montrons que si X et Y sont deux champs gaussiens à densités spectrales équivalentes, ils ont même régularité dans tout espace de Banach séparable s'injectant continument dans $\mathcal{C}^0(K)$ où K est un compact de $\mathbb{R}^d$.

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1 Introduction

In this note we are given two Gaussian random fields $\{X(x)\}_{x\in\mathbb{R}^d}$ and $\{Y(x)\}_{x\in\mathbb{R}^d}$ both admitting stationary increments. We also assume that these two fields

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admit a spectral density, that is there exists two positive functions $f_X, f_Y \in L^2(\mathbb{R}^d, (1 \wedge |\xi|^2)\mathrm{d}\xi)$ such that,

$$X(x) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}x.\xi} - 1) f_X^{1/2}(\xi) \mathrm{d}\widehat{W}(\xi) \ , \tag{1}$$

$$Y(x) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}x.\xi} - 1) f_Y^{1/2}(\xi) \mathrm{d}\widehat{W}(\xi) \ . \tag{2}$$

We are also given B a separable Banach space or a normed vector space, being the dual of a separable space. We assume that B is continuously embedded in $\mathcal{C}^0(K)$ where K denotes a compact of $\mathbb{R}^d$ (which is a separable Banach space). We aim at proving :

Theorem 1.1 *Assume that there exists some $C > 0$ such that*

$$f_X(\xi) \leq C f_Y(\xi) \text{ for all } \xi \in \mathbb{R}^d \ . \tag{3}$$

If the sample paths of $\{Y(x)\}_{x \in \mathbb{R}^d}$ a.s. belong to B then the sample paths of $\{X(x)\}_{x \in \mathbb{R}^d}$ a.s. belong to B.

2 Some classical results on probabilities in a Banach space

Here B_0 is a separable Banach space. We denote $\mathcal{B}_0$ the Borel $\sigma-$ algebra of B_0. If $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ is a probability space, a random element in $(B_0, \mathcal{B}_0)$ is a measurable mapping from $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ in $(B_0, \mathcal{B}_0)$.

Définition 2.1 *Let Z be a random element in $(B_0, \mathcal{B}_0)$ and $\mathbb{P}_Z$ its distribution. A distribution of regular conditional probability given Z is a mapping $f \in B_0 \mapsto \mathbb{P}(\cdot|Z = f)$ such that :*

(1) $\forall f \in B_0$, $\mathbb{P}(\cdot|Z = f)$ is a probability measure on $\mathcal{B}$.

(2) There exists a $\mathbb{P}_Z$-negligible set N such that

$$\forall f \in B_0 \setminus N, \mathbb{P}(\Omega \setminus Z^{-1}(f)|Z = f) = 0 \ .$$

(3) For all $A \in \mathcal{B}_\Omega$, the mapping $f \mapsto \mathbb{P}(A|Z = f)$ is $\mathbb{P}_Z$-measurable and

$$\mathbb{P}(A) = \int_{B_0} \mathbb{P}(A|Z = f) d\mathbb{P}_Z(f) \ .$$

In separable Banach spaces, the distribution of regular conditional probability given Z exists and is unique. More precisely :

Proposition 2.2 *For any random element Z in $(B_0, \mathcal{B}_0)$, there exists a distribution of conditional probability given Z, $f \mapsto \mathbb{P}(\cdot|Z = f)$. If $f \mapsto \widetilde{\mathbb{P}}(\cdot|Z = f)$ is another one, then the set $\{f, \mathbb{P}(\cdot|Z = f) \neq \widetilde{\mathbb{P}}(\cdot|Z = f)\}$ is negligible.*

Définition 2.3 *A random element X in $(B_0, \mathcal{B}_0)$ is Gaussian if, for any linear form $L \in B_0^*$ (where B_0^* denotes the dual space of B_0), $L(X)$ is a real Gaussian random variable.*

The independence of Gaussian random elements is characterized as follows [2] :

Proposition 2.4 *Two Gaussian random elements X_1 and X_2 in $(B_0, \mathcal{B}_0)$ are independent if for any linear forms L_1 and L_2 of B_0^*, one has*

$$\mathbb{E}(L_1(X_1)L_2(X_2)) = 0 \ .$$

3 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the following lemmas.

Lemma 3.1 *Let X and Y be two Gaussian fields of the form (1) and (2) with a.s. continuous sample paths. If $f_X \leq f_Y$ on $\mathbb{R}^d$, there exists two Gaussian fields X_1 and X_2 with stationary increments, independent as random elements with values in $\mathcal{C}^0(K)$, such that*

$$\{X(x)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \{X_1(x)\}_{x\in\mathbb{R}^d},\ \{Y(x)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \{X_1(x) + X_2(x)\}_{x\in\mathbb{R}^d} \ .$$

Proof. Let us consider the Gaussian random field Z defined on $\mathbb{R}^d \times \mathbb{R}^2$ by its covariance function

$$\begin{aligned} &\mathbb{E}(Z(x_1, \cdots, x_d; y_1, y_2)Z(x'_1, \cdots, x'_d; y'_1, y'_2)) \\ &= y_1y'_1 \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}x.\xi} - 1)(\mathrm{e}^{-\mathrm{i}x'.\xi} - 1) f_X(\xi)\mathrm{d}\xi + y_2y'_2 \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}x.\xi} - 1)(\mathrm{e}^{-\mathrm{i}x'.\xi} - 1)(f_Y(\xi) - f_X(\xi))\mathrm{d}\xi \ . \end{aligned}$$

The inequality $f_Y - f_X \geq 0$ on $\mathbb{R}^d$ implies that

$$((x_1, \cdots, x_d; y_1, y_2), (x'_1, \cdots, x'_d; y'_1, y'_2)) \mapsto \mathbb{E}(Z(x_1, \cdots, x_d; y_1, y_2)Z(x'_1, \cdots, x'_d; y'_1, y'_2)) \ ,$$

is positive definite. Set now for any $x \in \mathbb{R}^d$, $X_1(x) = Z(x; 1, 0)$ and $X_2(x) = Z(x; 0, 1)$. Hence one has

$$\{X(x)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \{X_1(x)\}_{x\in\mathbb{R}^d} \text{ and } \{Y(x)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \{Z(x, 1, 1)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \{X_1(x)+X_2(x)\}_{x\in\mathbb{R}^d} \ .$$

Moreover, for all x, x' in $\mathbb{R}^d$, $\mathbb{E}(X_1(x)X_2(x')) = 0$. Using Proposition 2.4 and a Fubini theorem, since the dual of $\mathcal{C}^0(B(0,1))$ is the set of Radon measures, this last equality implies that $\{X_1(x)\}_{x\in\mathbb{R}^d}$ and $\{X_2(x)\}_{x\in\mathbb{R}^d}$ are independent.

The assumptions and notations are now those of Theorem 1.1. The next lemma is a reformulation of the Anderson inequality (see Theorem 11.9 of [1]) :

Lemma 3.2 *Let $\{X(x)\}_{x\in K}$ a Gaussian random field defined on K with a.s. continuous sample paths. Then, for any $r > 0$ and $f \in \mathcal{C}^0(K)$*

$$\mathbb{P}(\|X + f\|_B \leq r) \leq \mathbb{P}(\|X\|_B \leq r) \ .$$

Proof. Consider X as a Gaussian random element in $B_0 = \mathcal{C}^0(K)$ which is a separable locally convex space. Observe that in Theorem 9 of [1] the set C need only to be a convex, symmetric Borelian set of B_0 (personal communication of M. Lifshits). Hence, we can apply Theorem 9 of [1] to the Gaussian measure $\mathbb{P}_X$ and to the set $C = \{g \in B, \ \|g\|_B \leq r\}$ which is convex (since $\|\cdot\|_B$ is a norm), closed in B since B is either Banach either the dual of a Banach space and then a Borelian of $B_0 = \mathcal{C}^0(K)$, and symmetric.

The following result can be deduced from Lemma 3.2 :

Lemma 3.3 *Let $\{X_1(x)\}_{x\in K}$ and $\{X_2(x)\}_{x\in K}$ two independent Gaussian random fields defined on a compact subset K of $\mathbb{R}^d$ with a.s. continuous sample paths. For any $r > 0$, one has*

$$\mathbb{P}(\|X_1 + X_2\|_B \leq r) \leq \mathbb{P}(\|X_1\|_B \leq r) \ .$$

Proof. Since X_1 and X_2 are independent as random elements in $\mathcal{C}^0(K)$, by definition of the conditional probability, one has

$$\mathbb{P}(\|X_1+X_2\|_B \leq r) = \int \mathbb{P}(\|X_1+f\|_B \leq r | X_2 = f) \mathrm{d}\mathbb{P}_{X_2}(f) = \int \mathbb{P}(\|X_1+f\|_B \leq r) \mathrm{d}\mathbb{P}_{X_2}(f) \ .$$

Lemma 3.1 applied to $X = X_1$ then implies that for any $f \in B$, $\mathbb{P}(\|X_1+f\|_B \leq r) \leq \mathbb{P}(\|X_1\|_B \leq r)$. Hence $\mathbb{P}(\|X_1 + X_2\|_B \leq r) \leq \mathbb{P}(\|X_1\|_B \leq r)$.

Theorem 1.1 follows from these lemmas since the assumption $f_X \leq C f_Y$ and Lemma 3.1 imply that

$$\{Y(x)\}_{x\in\mathbb{R}^d} \overset{(\mathcal{L})}{=} \left\{ \frac{1}{C^{1/2}} X(x) + X_2(x) \right\}_{x\in\mathbb{R}^d} \ .$$

Lemma 3.3 then yields the required result.

References

[1] M.A. Lifshits, *Gaussian Random Functions*, Kluwer Academic Publishers, 1995.

[2] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, 1967.